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A general scheme for the construction of minimum uncertainty coherent states of anharmonic oscillators

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Abstract

A mixed supersymmetric–algebraic approach to the construction of the minimum uncertainty coherent states of anharmonic oscillators is presented. It permits generating not only the well-known coherent states of the harmonic and Morse oscillators but also the so far unknown coherent states of the Wei Hua and generalized Morse and Kratzer–Fues oscillators. The method can be applied to generate superpotentials indispensable for deriving the Schrödinger equation in the supersymmetric form amenable to direct solution in the SUSYQM scheme.

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1. Introduction

The coherent states introduced by Schrödinger in 1926 [1] make a very useful tool for the investigation of various problems in quantum optics [2], in particular the interactions of matter with coherent radiation [3], for example the resonant interactions of the laser beam with molecules producing the coherent effects such as self-induced transparency, soliton formation [4], excitation of a coherent superposition of states [5] and periodic alternations of the refractive index in the molecular systems [5, 6]. In the latter case, the variation of the refractive index may appear due to the interaction of the coherent radiation with the coherent rotational [7] and pure vibrational [8] states of the molecules. Studies in this area require construction of the coherent states for anharmonic oscillators. Such states are defined in a similar manner to the ordinary coherent states of the harmonic oscillator [9]: (i) they are eigenstates of the annihilation operator, (ii) they minimize the generalized position–momentum uncertainty relation and (iii) they arise from the operation of a unitary displacement operator to the ground state of the oscillator. It should be pointed out that definition (iii) relies on the form of the displacement operator, which is specific to the harmonic oscillator [10], hence in this case mainly approximate coherent states can be derived using, for example, Nieto–Simons [11] or Kais–Levine [12] procedures. The point (ii) defines the so-called *intelligent* coherent states

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[13]; they not only minimize the Heisenberg uncertainty relation but also maintain this relation in time due to its temporal stability. The coherent states according to definition (i) are often called Barut–Girardello states [14].

Coherent states of anharmonic oscillators have been constructed using several alternative approaches. In the method proposed by Nieto and Simmons [11], the position and momentum operators are chosen in such a way that the resultant Hamiltonian resembles that for a harmonic oscillator. The coherent states are then determined on condition that they minimize the generalized uncertainty relation in the new variables. Perelomov [15] has derived the coherent states using the irreducible representations of a Lie group. This method has been successfully applied to generate the coherent states of the Morse [16] oscillator [12, 17]. The generalized coherent states can also be constructed using an algebraic method [10] and supersymmetric quantum mechanics (SUSYQM) [18], especially in the scheme employing the shape invariance introduced by Gendenshtein [19] and then developed by other authors [20–24]. Applying the above formalisms, the coherent states for Morse [10–12, 17, 25], Pöschl–Teller [26], hydrogen atom [27, 28], Eckart and Rosen–Morse [23], double-well and linear (gravitational) potentials [29] have been constructed.

In the present study, we introduce the mixed supersymmetric–algebraic method, which does not refer to the concept of shape invariance [19]. It permits the generation of not only the coherent states of the harmonic and Morse oscillators but also the so far unknown coherent states of the Wei Hua [30] and generalized Morse and Kratzer–Fues [31] oscillators.

2. The supersymmetric–algebraic method

The method starts from the vibrational dimensionless Schrödinger equation

$$\left[\frac{1}{2}\hat{p}^2 + V(q) - E_0\right]|v\rangle = \Delta E_{v0}|v\rangle, \qquad \hat{p} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}q}, \tag{1}$$

in which $\Delta E_{v0} = E_v - E_0$ whereas $q = u_r r$ denotes a dimensionless spatial variable r, with a scaling factor u_r depending on the explicit form of the potential energy term $V(q)$.

The vital point for the approach proposed is the assumption that the last two terms in the operator part of equation (1) can be specified in the form of the Riccati equation

$$V(q) - E_0 = \frac{1}{2}\left[x^2(q) + \frac{\mathrm{d}x(q)}{\mathrm{d}q}\right] \tag{2}$$

familiar in SUSYQM [33]. Here, $x(q)$ is the anharmonic coordinate, which satisfies the commutation relation $[x(q), \hat{p}] = \mathrm{i}\,\mathrm{d}x(q)/\mathrm{d}q$. Its form depends on the oscillator type, hence the explicit expression for $x(q)$ will be determined for a given form of the potential energy function. In SUSYQM $x(q)$ (with accuracy to sign) is interpreted as a superpotential [33], which permits the construction of the supersymmetric Schrödinger equation straightforward to analytical solutions.

Substituting equation (2) into (1), one gets the latter in the factorized form

$$\hat{A}^\dagger\hat{A}|v\rangle = \Delta E_{v0}|v\rangle, \tag{3}$$

in which

$$\hat{A} = \frac{1}{\sqrt{2}}\left[\frac{\mathrm{d}}{\mathrm{d}q} - x(q)\right], \qquad \hat{A}^\dagger = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}q} - x(q)\right], \qquad [\hat{A}, \hat{A}^\dagger] = -\frac{\mathrm{d}x(q)}{\mathrm{d}q}. \tag{4}$$

In order to construct the coherent state for the potential $V(q)$, we need a ground-state solution $|0\rangle$ of equation (3), which is an eigenstate of the operator $\hat{A}$. If $\hat{A}$ annihilates the ground state

$\hat{A}|0\rangle = 0$, then the coherent states $|\alpha\rangle$ are the eigenstates of the annihilation operator $\hat{A}$, and the following relations are fulfilled:

$$\hat{A}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \langle\alpha|\alpha^* = \langle\alpha|\hat{A}^\dagger, \qquad |\alpha\rangle = |0\rangle \exp(\sqrt{2}\alpha q). \tag{5}$$

The ground-state eigenfunction $|0\rangle$ appearing in (5) can be calculated by the integration of the annihilation equation $\hat{A}|0\rangle = 0$ yielding

$$|0\rangle = \exp\left[\int^q x(q')\,\mathrm{d}q'\right]. \tag{6}$$

The approach proposed adopts only part of the basic concepts of the SUSYQM as Riccati equation (2), superpotential $x(q)$ and its connection with the ground-state eigenfunction (6). However, it does not take into account the partner Hamiltonian $\hat{H}_- = \hat{A}\hat{A}^\dagger$ isospectral with $\hat{H}_+ = \hat{A}^\dagger\hat{A}$, which satisfies the shape-invariant condition [20–24],

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1), \tag{7}$$

in which a_1, a_2 are parameters that specify space-independent properties of the potentials such as strength, range and diffuseness; a_2 is a function of a_1 , and the remainder $R(a_1)$ is independent of the q-variable.

3. The minimum uncertainty coherent states

One may prove that the states $|\alpha\rangle$ minimize the generalized position–momentum uncertainty relation [10]

$$\begin{aligned} &(\Delta x(q))^2(\Delta p)^2 \geqslant \frac{1}{4}\langle\alpha|g(q)|\alpha\rangle^2, \\ &g(x) = -\mathrm{i}\,[x(q), \hat{p}] = \frac{\mathrm{d}x(q)}{\mathrm{d}q} = -[\hat{A}, \hat{A}^\dagger]. \end{aligned} \tag{8}$$

To prove this thesis let us calculate

$$\langle\alpha|x(q)|\alpha\rangle = \frac{1}{\sqrt{2}}\langle\alpha|\hat{A} + \hat{A}^\dagger|\alpha\rangle = \frac{1}{\sqrt{2}}(\alpha + \alpha^*), \tag{9}$$

$$\langle\alpha|\hat{p}|\alpha\rangle = -\mathrm{i}\frac{1}{\sqrt{2}}\langle\alpha|\hat{A} - \hat{A}^\dagger|\alpha\rangle = -\mathrm{i}\frac{1}{\sqrt{2}}(\alpha - \alpha^*), \tag{10}$$

$$\begin{aligned} 2\langle\alpha|x(q)^2|\alpha\rangle &= \langle\alpha|\hat{A}\hat{A} + 2\hat{A}^\dagger\hat{A} + \hat{A}^\dagger\hat{A}^\dagger - \frac{\mathrm{d}x(q)}{\mathrm{d}q}|\alpha\rangle \\ &= \left[(\alpha + \alpha^*)^2 - \langle\alpha|\frac{\mathrm{d}x(q)}{\mathrm{d}q}|\alpha\rangle\right], \end{aligned} \tag{11}$$

$$\begin{aligned} -2\langle\alpha|\hat{p}^2|\alpha\rangle &= \langle\alpha|\hat{A}\hat{A} - 2\hat{A}^\dagger\hat{A} + \hat{A}^\dagger\hat{A}^\dagger + \frac{\mathrm{d}x(q)}{\mathrm{d}q}|\alpha\rangle \\ &= \left[(\alpha - \alpha^*)^2 + \langle\alpha|\frac{\mathrm{d}x(q)}{\mathrm{d}q}|\alpha\rangle\right], \end{aligned} \tag{12}$$

in which equation (4) is employed in the form $\hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A} - \mathrm{d}x(q)/\mathrm{d}q$.

Having calculated (9)–(12), we can pass to evaluate

$$(\Delta x(q))^2 = \langle\alpha|x(q)^2|\alpha\rangle - \langle\alpha|x(q)|\alpha\rangle^2 = -\frac{1}{2}\langle\alpha|\frac{\mathrm{d}x(q)}{\mathrm{d}q}|\alpha\rangle, \tag{13}$$

$$(\Delta p)^2 = \langle\alpha|\hat{p}^2|\alpha\rangle - \langle\alpha|\hat{p}|\alpha\rangle^2 = -\frac{1}{2}\langle\alpha|\frac{\mathrm{d}x(q)}{\mathrm{d}q}|\alpha\rangle, \tag{14}$$

providing $\Delta x(q) = \Delta p$ and

$$(\Delta x(q))^2(\Delta p)^2 = \frac{1}{4}\langle\alpha|\frac{\mathrm{d}x(q)}{\mathrm{d}q}|\alpha\rangle^2. \tag{15}$$

The calculations performed prove that the states $|\alpha\rangle$ minimize the generalized position–momentum uncertainty relation for the anharmonic coordinate $x(q)$. They are also the eigenstates of the operator $\hat{A}$, which annihilates the ground state $\hat{A}|0\rangle = 0$, hence they satisfy the two fundamental requirements established for the coherent states of an anharmonic oscillator.

4. The harmonic oscillator

In order to demonstrate how the method works, let us calculate first the well-known coherent states of the harmonic oscillator. To this purpose let us assume that

$$\frac{\mathrm{d}x(q)}{\mathrm{d}q} = -1 \Longrightarrow x(q) = -q \qquad \text{for} \quad x(0) = 0. \tag{16}$$

Then equation (1) including (2) turns out to be the well-known Schrödinger equation for the ground state $|0\rangle = \exp(-q^2/2)$ of a harmonic oscillator, whereas the operators $\hat{A}$ and $\hat{A}^\dagger$ take the well-known form of annihilation and creation operators, which satisfy the commutation rule $[\hat{A}, \hat{A}^\dagger] = -\mathrm{d}x(q)/\mathrm{d}q = 1$. Hence, the coherent states of a harmonic oscillator can be specified by the general formula (5):

$$|\alpha\rangle = |0\rangle \exp(\sqrt{2}\alpha q) = \exp\left(-\tfrac{1}{2}q^2\right)\exp(\sqrt{2}\alpha q). \tag{17}$$

5. The generating function

The results obtained for a harmonic oscillator indicate that crucial for the method proposed is the explicit form of the term $\mathrm{d}x(q)/\mathrm{d}q$, which in the general case can be given as a negative x-dependent function

$$\frac{\mathrm{d}x(q)}{\mathrm{d}q} = -f(x). \tag{18}$$

Hence, employing different analytical functions $f(x)$ one may generate a variety of potentials and associated superpotentials satisfying relation (2). The indispensable for this purpose $x(q)$ can be calculated from (18) by integration, provided that we known the explicit form of $f(x)$. Assuming that $|x(q)| < 1$ one may expand $f(x)$ in a power series

$$f(x) = c_1(x + c_0/c_1) + c_2(x + c_0/c_1)^2 + \cdots, \tag{19}$$

and then successively apply the first-, second- and higher-order terms in the determination of coherent states of anharmonic potentials.

6. The Morse oscillator

In the simplest case of the linear expansion

$$f(x) = c_1(x + c_0/c_1), \tag{20}$$

equation (18) and the initial condition $x(0) = (1 - c_0)/c_1$ provide

$$x(q) = \frac{1}{c_1}[\exp(-c_1 q) - c_0]. \tag{21}$$

Introducing (20) into (2) and (6) one gets the Schrödinger equation

$$\frac{1}{2}\left\{-\frac{d^2}{dq^2} + \frac{1}{c_1^2}[c_0 + c_1^2/2 - \exp(-c_1 q)]^2 - c_0 - \frac{c_1^2}{4}\right\}|0\rangle = 0 \tag{22}$$

and the ground-state solution

$$|0\rangle = \exp\left[-\frac{1}{c_1^2}\exp(-c_1 q)\right]\exp\left[-(c_0/c_1)q\right]. \tag{23}$$

Redefining the constants $c_0 = s - x_e$, $c_1 = \sqrt{2x_e}$ one may transform equations (22) and (23) into the form

$$\frac{1}{2}\left\{-\frac{d^2}{dq^2} + \frac{1}{2x_e}[s - \exp(-\sqrt{2x_e}q)]^2 - s + \frac{x_e}{2}\right\}|0\rangle = 0 \tag{24}$$

$$|0\rangle = \exp\left[-\frac{1}{2x_e}\exp(-\sqrt{2x_e}q)\right]\exp\left[-\frac{(s - x_e)q}{\sqrt{2x_e}}\right], \tag{25}$$

describing the generalized quantum Morse oscillator [16] endowed with the potential $V(r) = D_e[s - \exp(-ar)]^2$ and the ground-state energy $E_0 = s/2 - x_e/4$ as the special case of the general formula

$$E_v = s\left(v + \frac{1}{2}\right) - x_e\left(v + \frac{1}{2}\right)^2. \tag{26}$$

For $s = 1$ the specified above formulae reduce to the well-known equations derived by Cooper [10], in which $x_e = \hbar\omega_e/(4D_e)$ denotes the anharmonicity constant, $\omega_e = a\sqrt{2D_e/m}$ is the vibrational frequency defined by the reduced mass m of the system, with the dissociation energy D_e and the range parameter a appearing in the Morse potential.

Equations (20) and (4) produce the generalized Morse annihilation and creation operators

$$\hat{A} = \frac{1}{\sqrt{2}}\left[\frac{d}{dq} + \frac{s - \exp(-\sqrt{2x_e}q)}{\sqrt{2x_e}} - \sqrt{\frac{x_e}{2}}\right],$$
$$\hat{A}^\dagger = \frac{1}{\sqrt{2}}\left[-\frac{d}{dq} + \frac{s - \exp(-\sqrt{2x_e}q)}{\sqrt{2x_e}} - \sqrt{\frac{x_e}{2}}\right] \tag{27}$$

and the associated coherent states

$$|\alpha\rangle = \exp\left[-\frac{1}{2x_e}\exp(-\sqrt{2x_e}q)\right]\exp\left[-\frac{(s - x_e)q}{\sqrt{2x_e}}\right]\exp(\sqrt{2}\alpha q). \tag{28}$$

They are eigenstates of the annihilation operator $\hat{A}$, which minimize the uncertainty relation (8) for $[\hat{A}, \hat{A}^\dagger] = \exp(-\sqrt{2x_e}q)$.

7. The Wei Hua oscillator

Taking into account the parabolic expansion

$$f(x) = c_1(x + c_0/c_1) + c_2(x + c_0/c_1)^2 \tag{29}$$

and the identical initial condition as previously $x(0) = (1 - c_0)/c_1$, from equation (18), one obtains

$$x(q) = \frac{(cc_1/c_2)\exp[-c_1(q - q_0)]}{1 - c\exp[-c_1(q - q_0)]} - \frac{c_0}{c_1}, \tag{30}$$

in which $c = C/(B/W + C)$, $q_0 = \ln(B/W + C)/c_1$, $W = \left(2c_0 + c_1^2\right)\big/[2c_1(1 - c_2)]$, $B = c_1\big/\left(c_1^2 + c_2\right)$ and $C = c_2\big/\left(c_1^2 + c_2\right)$. Hence, the Schrödinger equation (3) and its ground-state solution take the forms

$$\frac{1}{2}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + 2D\left\{\frac{1 - \exp[-c_1(q - q_0)]}{1 - c\exp[-c_1(q - q_0)]}\right\}^2 - 2E_0\right\}|0\rangle = 0, \tag{31}$$

$$|0\rangle = \{1 - c\exp[-c_1(q - q_0)]\}^{\frac{1}{c_2}}\{c\exp[-c_1(q - q_0)]\}^{\frac{c_0}{c_1^2}}, \tag{32}$$

in which

$$2D = (1 - c_2)W^2, \qquad 2E_0 = (1 - c_2)W^2 - c_0^2\big/c_1^2 = 2D - c_0^2\big/c_1^2 \tag{33}$$

represent the dissociation energy and ground eigenenergy of the system. Equation (31) is the well-known eigenvalue equation for the ground state of the Wei Hua oscillator [30], whose coherent states have not been derived as yet. Applying equations (4), (30) and (32) one gets the annihilation and creation operators

$$\hat{A} = \frac{1}{\sqrt{2}}\left[\frac{\mathrm{d}}{\mathrm{d}q} - \frac{(cc_1/c_2)\exp[-c_1(q - q_0)]}{1 - c\exp[-c_1(q - q_0)]} + \frac{c_0}{c_1}\right], \tag{34}$$

$$\hat{A}^\dagger = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}q} - \frac{(cc_1/c_2)\exp[-c_1(q - q_0)]}{1 - c\exp[-c_1(q - q_0)]} + \frac{c_0}{c_1}\right], \tag{35}$$

as well as the coherent states of the Wei Hua oscillator

$$|\alpha\rangle = \{1 - c\exp[-c_1(q - q_0)]\}^{\frac{1}{c_2}}\{c\exp[-c_1(q - q_0)]\}^{\frac{c_0}{c_1^2}}\exp(\sqrt{2}\alpha q). \tag{36}$$

They are eigenstates of the annihilation operator $\hat{A}|\alpha\rangle = \alpha|\alpha\rangle$ and minimize the uncertainty relation (8) for $[\hat{A}, \hat{A}^\dagger] = \left(cc_1^2/c_2\right)\exp[-c_1(q - q_0)]/\{1 - c\exp[-c_1(q - q_0)]\}^2$.

Employing the parameter relations

$$\frac{c_0}{c_1^2} = \rho_0, \qquad \frac{1}{c_2} = \rho + \frac{1}{2}, \qquad c_1 = b, \tag{37}$$

in which [30]

$$\rho_0 = (t^2 - \lambda_0)^{1/2}, \qquad \rho = \left[\frac{1}{4} + t^2(Q - 1)^2\right]^{1/2}, \qquad Q = \frac{1}{c}, \tag{38}$$

$$t^2 = \frac{2D}{b^2}, \qquad 2E_0 = b^2\lambda_0, \qquad \hbar = m = 1, \tag{39}$$

the derived ground-state eigenfunction (32) can be written in the form originally obtained by Wei Hua [30]

$$|0\rangle = \{1 - c\exp[-b(q - q_0)]\}^{\rho + \frac{1}{2}}\{c\exp[-b(q - q_0)]\}^{\rho_0}, \tag{40}$$

whereas from equations (37)–(39) one gets

$$2E_0 = b^2\lambda_0 = b^2\left(t^2 - \rho_0^2\right) = 2D - b^2\rho_0^2 = (1 - c_2)W^2 - c_0^2\big/c_1^2 \tag{41}$$

in agreement with the ground-state eigenvalue formula (33). Employing the notation (38), the annihilation and creation operators as well as the coherent states of the Wei Hua oscillator can be specified in the form

$$\hat{A} = \frac{1}{\sqrt{2}}\left[\frac{\mathrm{d}}{\mathrm{d}q} - \frac{bc\left(\rho+\frac{1}{2}\right)\exp[-b(q-q_0)]}{1-c\exp[-b(q-q_0)]} + b\rho_0\right], \tag{42}$$

$$\hat{A}^{\dagger} = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}q} - \frac{bc\left(\rho+\frac{1}{2}\right)\exp[-b(q-q_0)]}{1-c\exp[-b(q-q_0)]} + b\rho_0\right], \tag{43}$$

$$|\alpha\rangle = \{1-c\exp[-b(q-q_0)]\}^{\rho+\frac{1}{2}}\{c\exp[-b(q-q_0)]\}^{\rho_0}\exp(\sqrt{2}\alpha q). \tag{44}$$

8. The Kratzer–Fues oscillator

The method proposed permits also a derivation of the coherent states of the vibrational systems described by the Kratzer–Fues potential [31, 32]

$$V(r) = D_e\left[\frac{r-r_e}{r}\right]^2 = D_e\left[\frac{z}{1+z}\right]^2 = D_e\left[\frac{y-1}{y}\right]^2. \tag{45}$$

Here r_e denotes the equilibrium length of the molecule bond $V(r_e) = 0$, $D_e = V(r \to \infty)$ is the potential depth approximately equal to the dissociation energy of the system, whereas $z = (r - r_e)/r_e$ and $y = r/r_e$ are the Dunham and Kratzer–Fues variables, respectively.

In order to construct the coherent states of the Kratzer–Fues oscillator, we employ the generating function

$$f(x) = [c_1(x + c_0/c_1)]^2, \tag{46}$$

which when introduced into equation (18) yields

$$x(q) = \frac{1}{c_1(c_1q+1)} - \frac{c_0}{c_1}, \qquad x(0) = (1-c_0)/c_1. \tag{47}$$

Hence, the Schrödinger equation (1) and its ground-state solution take the form

$$\frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + 2D\left(\frac{c_1q-s}{1+c_1q}\right)^2 - 2E_0\right]|0\rangle = 0, \tag{48}$$

$$|0\rangle = (1+c_1q)^{\frac{1}{c_1^2}}\exp\left[-\frac{\left(1-c_1^2\right)(c_1q+1)}{c_1^2(s+1)}\right], \tag{49}$$

in which $2D = c_0^2/\left[c_1^2\left(1-c_1^2\right)\right]$, $s = \left(1-c_0-c_1^2\right)/c_0$, $2E_0 = c_0^2/\left(1-c_1^2\right)$. It is easy to verify that for $s = 0$ or $c_0 = 1 - c_1^2$ and $c_1q = z$, equations (48) and (49) turn out to be the Schrödinger equation for the Kratzer–Fues oscillator [31]

$$\frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + 2D\left(\frac{c_1q}{1+c_1q}\right)^2 - 2E_0\right]|0\rangle = 0, \tag{50}$$

$$|0\rangle = (1+c_1q)^{\frac{1}{c_1^2}}\exp\left[-\left(1-c_1^2\right)\left(c_1q+1\right)/c_1^2\right], \tag{51}$$

in which $2D = \left(1-c_1^2\right)/c_1^2$, $2E_0 = 1-c_1^2$. Hence, the Kratzer–Fues annihilation and creation operators can be given in the form

$$\hat{A} = \frac{1}{\sqrt{2}}\left[\frac{\mathrm{d}}{\mathrm{d}q} - \frac{1}{c_1(c_1q+1)} + \frac{1-c_1^2}{c_1}\right],$$
$$\hat{A}^{\dagger} = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}q} - \frac{1}{c_1(c_1q+1)} + \frac{1-c_1^2}{c_1}\right], \tag{52}$$

whereas the associated coherent states are

$$|\alpha\rangle = (1 + c_1 q)^{\frac{1}{c_1^2}} \exp\left[-\left(1 - c_1^2\right)(c_1 q + 1)/c_1^2\right] \exp(\sqrt{2}\alpha q). \tag{53}$$

Such states minimize the uncertainty relation (8) for $[\hat{A}, \hat{A}^\dagger] = (c_1 q + 1)^{-2}$.

Taking into account the parameter relationships ($\hbar = m = 1$)

$$\frac{1}{c_1^2} = \lambda, \qquad \frac{1 - c_1^2}{c_1^4} = \gamma^2 = 2r_e^2 D_e \quad \beta_0^2 = 2r_e^2(D_e - E_0) \quad \gamma^2/\lambda = \beta_0 \tag{54}$$

and the Kratzer–Fues variable $y = c_1 q + 1$, we can rewrite equations (50) and (52) to the form recently derived by Molski [7]

$$\hat{A} = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}y} + \beta_0 - \frac{\lambda}{y}\right), \qquad \hat{A}^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}y} + \beta_0 - \frac{\lambda}{y}\right), \tag{55}$$

$$\hat{A}|\alpha\rangle = \alpha|\alpha\rangle, \qquad |\alpha\rangle = y^\lambda \exp[-\beta_0 y] \exp[\sqrt{2}\alpha y]. \tag{56}$$

9. The generalized Kratzer–Fues oscillator

The term $D[(z - s)/(1 + z)]^2$, which appears in (48), is worth considering as it represents a generalized version of the Kratzer–Fues formula $V(r) = D_e[1 - r_e(s + 1)/r]^2$. From the relation $-1 \leqslant 1 - r_e(s + 1)/r \leqslant 1$, one may calculate the convergence radius $R \in [r_e(s + 1)/2, \infty]$ for the new potential. It increases for $s \in (-1, 0)$ in comparison with the radius of the original Kratzer–Fues potential ($s = 0$) yielding $R \in (r_e/2, \infty)$. In such circumstances, the expansion of the potential energy function

$$V(r) = n_0\left[\frac{r - r_e(s + 1)}{r}\right]^2 \left\{1 + \sum_{k=1}^{N} n_k \left[\frac{r - r_e(s + 1)}{r}\right]^k\right\} \tag{57}$$

into a series of the generalized Kratzer–Fues variable $1 - r_e(s + 1)/r$ will provide much accurate reproduction of the real potential curves than that obtained by the Simons–Parr–Finlan expansion ($s = 0$) [34], which diverges in the united-atom limit $r \to 0$. The set of parameters ($r_e, s, n_0, n_1, \ldots$) can be evaluated from the molecular IR and MW spectra by making use of the fitting procedure. It should also be pointed out that the new potential permits the analytical solution of the Schrödinger equation and can be used to generate the coherent states for the generalized Kratzer–Fues oscillator:

$$|\alpha\rangle = (1 + c_1 q)^{\frac{1}{c_1^2}} \exp\left[-\left(1 - c_1^2\right)(c_1 q + 1)/c_1^2(s + 1)\right] \exp(\sqrt{2}\alpha q). \tag{58}$$

10. Conclusions

The method proposed is general and permits the construction of the coherent states, associated potentials and superpotentials as well as deriving the supersymmetric Schrödinger equation amenable to direct solution in the SUSYQM scheme [35]. The results of the calculations are presented in table 1. In the standard approach, the superpotentials are solutions of the Riccati equation (2) obtained for the specific form of the potential function $V(q)$ [35]. Here a new procedure has been introduced, which permits simultaneous derivation of the potentials and associated superpotentials assuming that $\mathrm{d}x(q)/\mathrm{d}q = -f(x)$. This term can be expanded in a power series of $x(q)$ and then used to generate the coherent states for different orders of the expansion (19). For the terms up to the second order, the method produces the minimum

Table 1. Comparison of the results obtained in this work for harmonic, generalized Morse, Wei Hua and generalized Kratzer–Fues oscillators; $f(x)$ is generating function; $x(q)$: superpotential; $V(q)$: potential. The parameters D, c and q_0 in the Wei Hua potential are defined in section 7.

$f(x)$	$x(q)$	$V(q)$
1	$-q$	q^2
$c_1\left(x+\frac{c_0}{c_1}\right)$	$\frac{\exp(-c_1 q)}{c_1}-\frac{c_0}{c_1}$	$\frac{\left[c_0+c_1^2/2-\exp(-c_1 q)\right]^2}{c_1^2}$
$c_1\left(x+\frac{c_0}{c_1}\right)+c_2\left(x+\frac{c_0}{c_1}\right)^2$	$\frac{(cc_1/c_2)\exp[-c_1(q-q_0)]}{1-c\exp[-c_1(q-q_0)]}-\frac{c_0}{c_1}$	$D\left\{\frac{1-\exp[-c_1(q-q_0)]}{1-c\exp[-c_1(q-q_0)]}\right\}^2$
$\left[c_1\left(x+\frac{c_0}{c_1}\right)\right]^2$	$\frac{1}{c_1(c_1 q+1)}-\frac{c_0}{c_1}$	$\left(\frac{c_0^2}{c_1^2(1-c_1^2)}\right)\left[\frac{c_1 q-(1-c_0-c_1^2)/c_0}{1+c_1 q}\right]^2$

uncertainty coherent states for harmonic, Morse, Wei Hua, Kratzer–Fues and generalized Morse and Kratzer–Fues potentials. They are the most important potential energy functions employed in molecular quantum mechanics, coherent spectroscopy (femtochemistry) and coherent nonlinear optics. In particular, they can be used in the investigation of the resonant interactions of the laser beam with molecules producing the coherent effects such as self-induced transparency, soliton formation, excitation of a coherent superposition of states and periodic alternations of the refractive index [4–8].

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